Picture Perfect Squares

May 13, 2009, 5:30 – 6:30 PM

A workshop presentation by:

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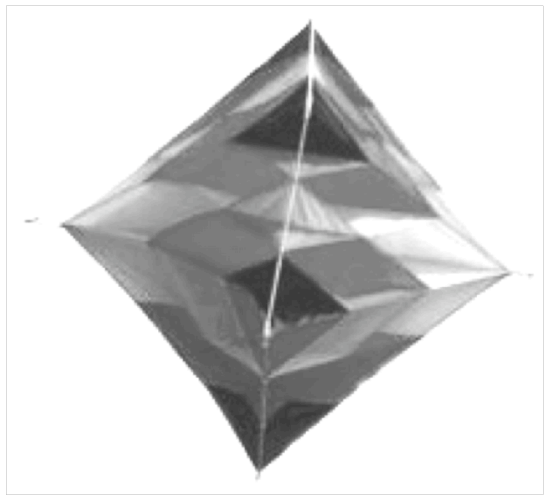


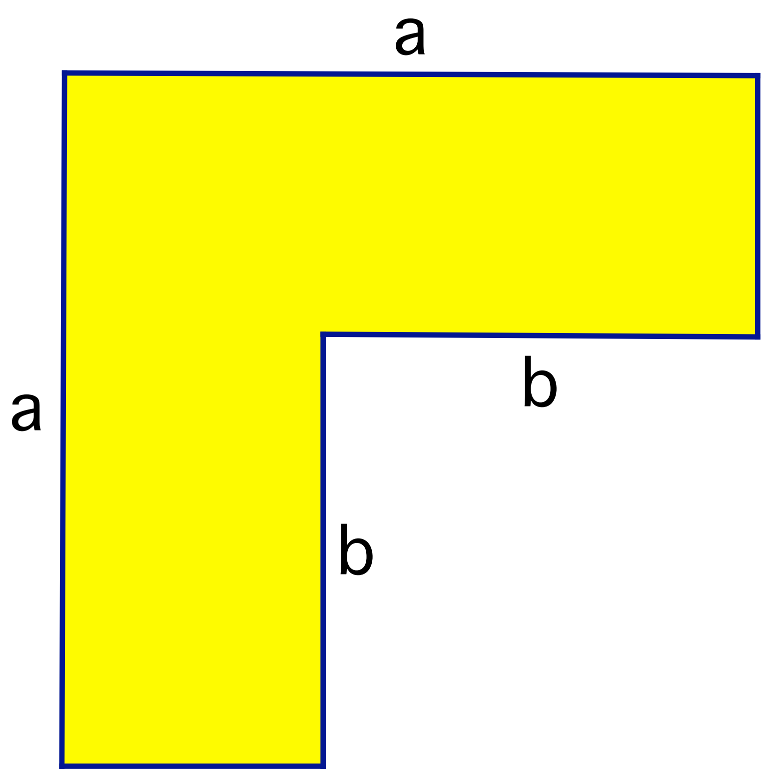
Rhode Island Mathematics Teachers Association

2009 Spring Meeting

May 13, 2009

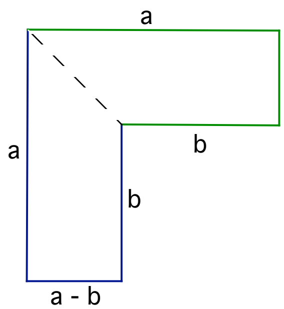
Quonset "O" Club, North Kingstown, RI

Find the area of the shaded (yellow) region algebraically **in at least three different ways**:



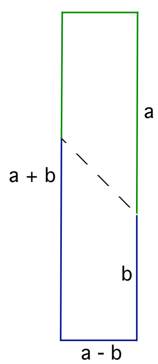
Solution 1:
Large square has area a^2, small inner square has area b^2.
The difference is $a^2 - b^2$

Solution 2:
Think of the shape being made of two congruent right trapezoids as follows:



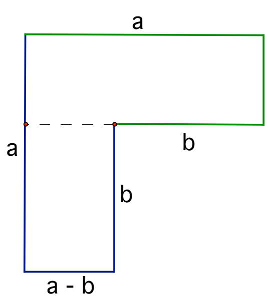
The height of each is $a - b$. Thus the total area is $2(\frac{1}{2}(a-b)(a+b)) = a^2 - b^2$

Solution 3:



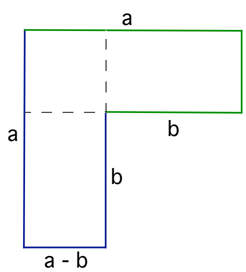
Take the green trapezoid, rotate it 90 degrees and reflect vertically. Then reconstruct the two trapezoids into a rectangle with dimensions $a + b$ and $a - b$. thus the area is $a^2 - b^2$.

Solution 4:



$$a(a-b) + b(a-b) =$$
$$(a+b)(a-b) =$$
$$a^2 - b^2$$

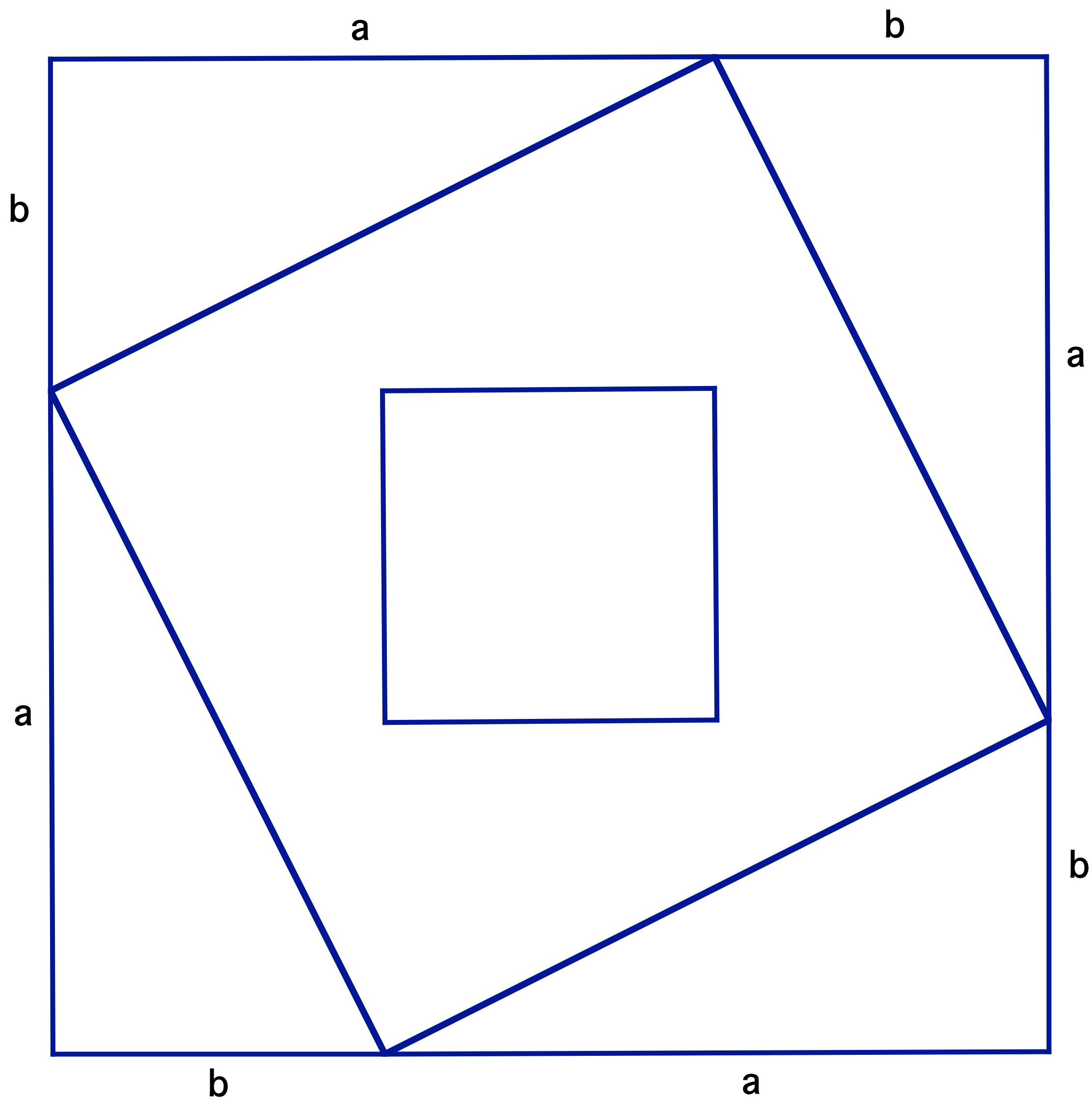
Solution 5:



$$2[b(a-b)] + (a-b)^2 =$$
$$2ba - 2b^2 + a^2 - 2ab + b^2 =$$
$$a^2 + 2ba - 2ab + b^2 - 2b^2 =$$
$$a^2 - b^2$$

The Perfect Square

Determine the area of each of the three squares in the figure below in terms of *a* and *b*. If possible, determine the areas in more than one way. What is the relationship between the three areas?



Show all your work:

Solutions:
An initial analysis of the figure (see figure 1) shows that the large square has a dimension of $a + b$; the medium square has a dimension of $\sqrt{a^2 + b^2}$; and the small square has a dimension of $a - b$. This then determines their respective areas to be: $(a+b)^2$, $a^2 + b^2$, and $(a-b)^2$. It can also be discovered both geometrically and algebraically that the difference between the areas of the large square and the medium square is the same as the difference between the areas of the medium square and the small square, namely $2ab$.

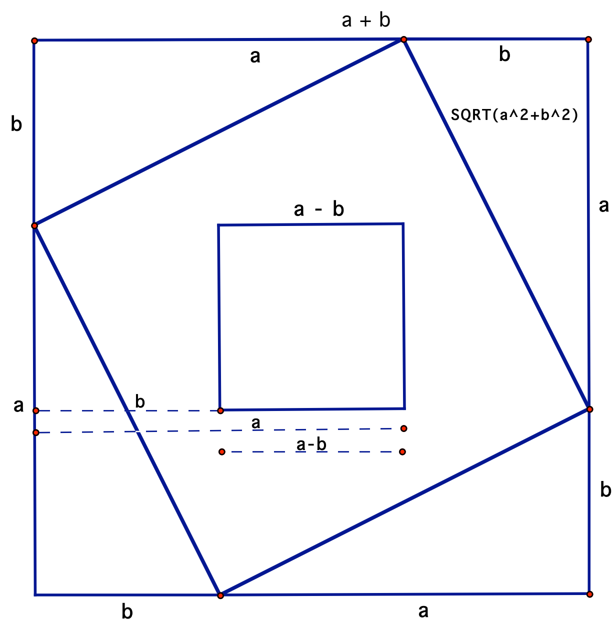


Figure 1

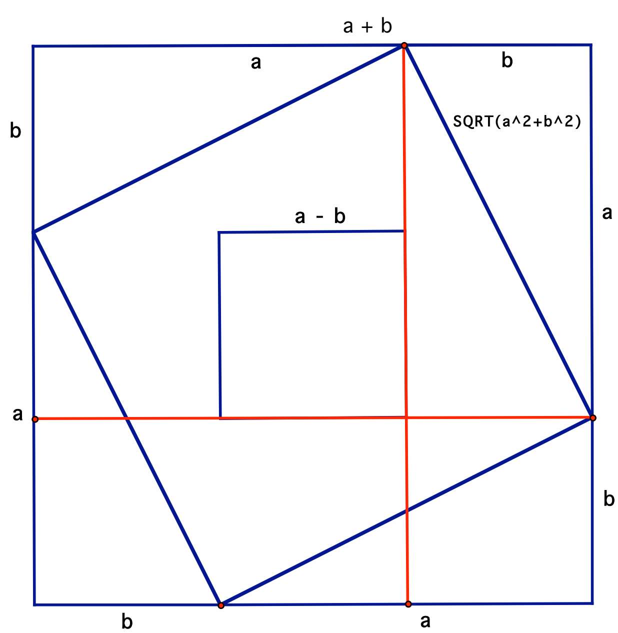


Figure 2

Let's look at some alternative geometric views and their algebraic translation of these areas. The area of the large square is algebraically straightforward by calculating the product of $(a+b)^2$. However in this figure the partial products are not directly visible. This is accomplished by drawing two segments as demonstrated in figure 2.

From the Pythagorean theorem we know that the side of the medium square is $\sqrt{a^2+b^2}$, thus the area of the medium square is $\left(\sqrt{a^2+b^2}\right)^2 = a^2+b^2$. Or for a visual representation see figure 3.

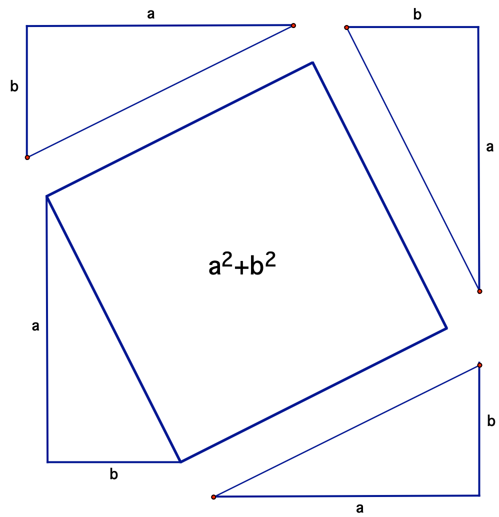


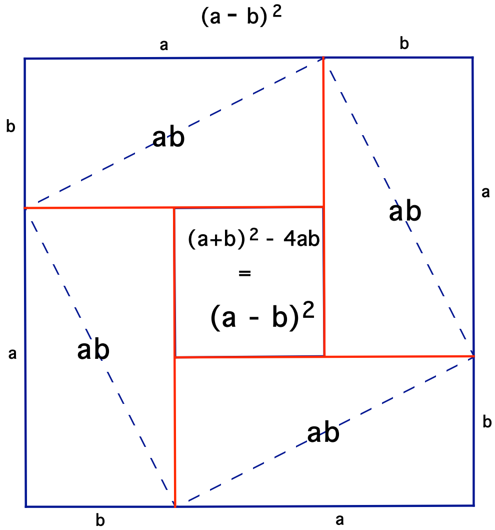
Figure 3.

Alternatively the area of the medium square can be seen as the large square less four times the area of a right triangle:

$$(a+b)^2 - 4(\tfrac{1}{2}ab) =$$
$$a^2 + 2ab + b^2 - 2ab =$$
$$a^2 + b^2$$

The side of the small square is a – b by construction (see figure 1). Therefore the area is:
$(a-b)^2 = a^2 - 2ab + b^2$
Another way to reason with the inner square is as follows:

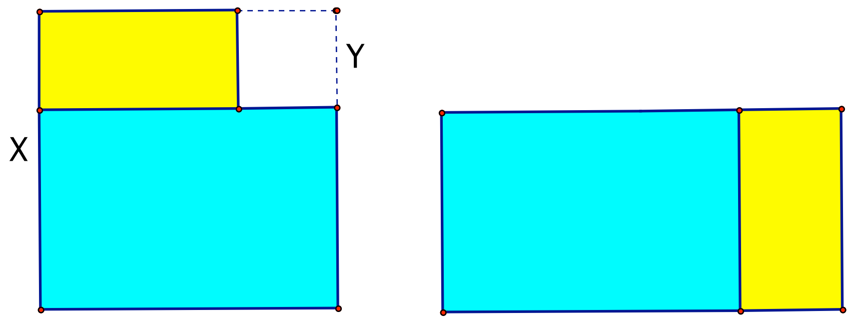
$$(a+b)^2 - 4ab =$$
$$a^2 + 2ab + b^2 - 4ab =$$
$$a^2 - 2ab + b^2 =$$
$$(a-b)^2$$



Alternatively we consider the area of the small square to be the area of the medium square less the area of the four right triangles:

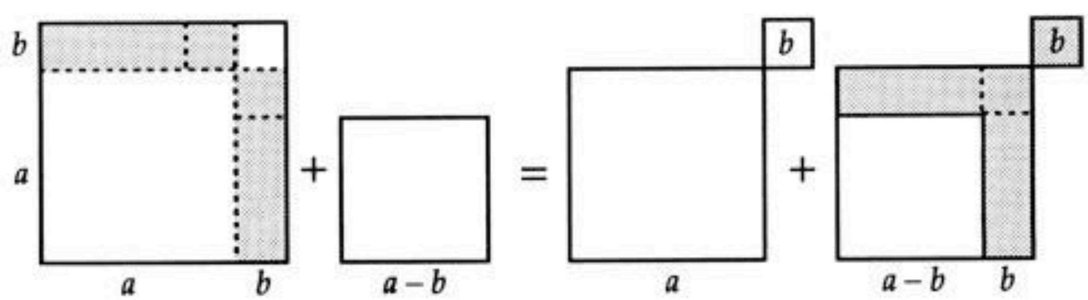
$$a^2 + b^2 - 4(\tfrac{1}{2}ab) =$$
$$a^2 - 2ab + b^2 =$$
$$(a-b)^2$$

$X^2 - Y^2 = (X+Y)(X-Y)$



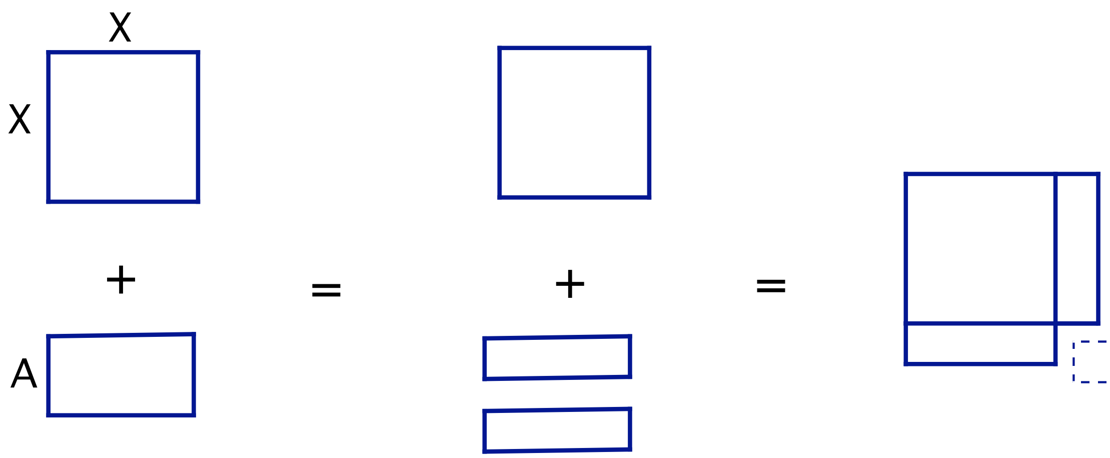
Algebraic Areas I

$$(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$$



Completing the Square:

$X^2 + AX = (X + \frac{1}{2}A)^2 + (\frac{1}{2}A)^2$



References and resources:

James Galant (1983). Proof without words. Mathematics Magazine, V56 (2), p.110.
Shirley Wakin (1984). Proof without words. *Mathematics Magazine,* V57 (4), p. 231.
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